On "Dotsenko-Fateev" representation of the toric conformal blocks

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Abstract

We demonstrate that the recent ansatz of [12], inspired by the original remark of R.Dijkgraaf and C.Vafa, reproduces the toric conformal blocks in the same sense that the spherical blocks are given by the integral representation of [1, 2] with a peculiar choice of open integration contours for screening insertions. In other words, we provide some evidence that the toric conformal blocks are reproduced by appropriate β-ensembles not only in the large-N limit, but also at finite N. The check is explicitly performed at the first two levels for the 1-point toric functions. Generalizations to higher genera are briefly discussed.

1 Introduction

In [1, 2, 3] we suggested a matrix model type (β-ensemble) representation of conformal blocks, which we called "Dotsenko-Fateev representation". It is based on the old idea of "conformal matrix models" [13, 14, 15] and it differs from the original Dotsenko-Fateev formulas for conformal blocks in minimal models [16, 17, 18] by substitution of the closed integration contours of screening operators by a carefully chosen open (non-closed) contours with a single (rather than two) screening operator. This integral representation of conformal blocks is believed to be helpful in study of the still mysterious AGT relation [19]-[55], and it has been further explored in a number of interesting papers [4]-[12].

In particular, in an very recent ref.[12] a generalization from spherical to toric conformal blocks was put forward in the large N limit, based on the original remark in [15]. Similar generalization at finite N, i.e, in the same sense as the spherical block was reproduced in refs.[1, 2, 3], was proposed in ref.[2], s.5, for a particular class of toric conformal blocks. The results of [15, 12] imply how this generalization can be done for the generic toric conformal block at finite N. The goal of the present paper is to demonstrate that the suggestion of [15, 12] indeed, works at finite N. We show explicitly that the 1-point toric function (conformal block)

$$B(q) = 1 + B_1 q + B_2 q^2 + \dots \tag{1}$$

with external dimension Δ_{ext}, internal dimension Δ, central charge c and coefficients

$$B_1 = \frac{\Delta_{ext}(\Delta_{ext}-1)}{2\Delta} + 1 \tag{2}$$

$$B_2 = \frac{1}{4\Delta(c + 2c\Delta - 10\Delta + 16\Delta^2)} \times \Big((8\Delta + c)\Delta_{ext}^4 + (-64\Delta - 2c)\Delta_{ext}^3 + (8c\Delta + 3c + 56\Delta + 128\Delta^2)\Delta_{ext}^2 +$$

$$+ (-2c - 8c\Delta - 128\Delta^2)\Delta_{ext} - 80\Delta^2 + 128\Delta^3 + 8c\Delta + 16c\Delta^2 \Big) \tag{3}$$

is reproduced, at least at the first two orders of q-expansion, by the generalized matrix model of the form (cf. [2, formula (63)])

$$Z_{DF}(q) = \int\limits_0^{2\pi} dz_1 \dots \int\limits_0^{2\pi} dz_N \prod_{i<j} \theta_*(z_i - z_j)^{2\beta} \prod_i \theta_*(z_i)^{2\mu} \prod_i e^{IAz_i} = \text{const} \cdot \Big(1 + J_1 q + J_2 q^2 + \dots \Big) \tag{4}$$

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where $\theta_*(z)$ is (the exponent of) the holomorphic Green function of free fields on a torus:

$$\theta_*(z) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin \frac{(2n+1)z}{2} = \sin \frac{z}{2} - q \sin \frac{3z}{2} + q^3 \sin \frac{5z}{2} + \dots \tag{5}$$

The partition function $Z_{DF}(q)$ can be viewed as a two-fold generalization of conventional eigenvalue matrix models: first, the ordinary differences $z_i - z_j$ are substituted by $\theta_*(z_i - z_j)$, second, they are raised to arbitrary powers β. These types of generalizations are often called elliptic- and β-deformations, respectively. For reader's convenience, we still use the term (generalized) matrix model for $Z_{DF}(q)$, instead of the more precise term "elliptic β-ensemble".

As explained in [2], in the spherical case such matrix models naturally appear as free-field correlators with insertions of N screening operators, integrated over their positions $z_1, \dots, z_N$. The main new ingredient for the torus (except for an obvious modification of the Green function) is introduction of an additional parameter A into the partition function. Our aim is to check, perturbatively at levels one and two, the following statement:

$$\boxed{\left(\prod_{n=1}^{\infty} (1 - q^n) \right)^{\nu} \times \left(1 + j_1 q + j_2 q^2 + \dots \right) = \left(1 + B_1 q + B_2 q^2 + \dots \right)} \tag{6}$$

Here $j_k = J_k\left(A, \mu = -\beta N, \beta, N\right)$ are the "on-shell" coefficients of the partition function, i.e. those taken at particular values of μ, satisfying the momentum conservation law $\mu + \beta N = 0$. We check the correspondence and find the precise relation between the parameters A, β, N of the Dotsenko-Fateev partition function and parameters Δ_{ext}, Δ, c of the conformal block (as well as the parameter ν, the power of the $U(1)$-factor). Our result in the 1-point sector is

$$\boxed{\begin{aligned} \Delta &= \frac{A^2 - (\beta - 1)^2}{4\beta}, \qquad \Delta_{ext} = \beta N^2 + \beta N - N \\[2mm] \nu &= 3\Delta_{ext} + 3N - 1 \\[2mm] c &= 1 - 6 \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}} \right)^2 \end{aligned}} \tag{7}$$

The present state of knowledge, therefore, can be summarized as follows. It is checked that Dotsenko-Fateev type β-ensembles with certain choices of integration contours describe the n-point spherical, and 1-point toric conformal blocks exactly (at finite N). A similar check for the n-point toric functions remains to be done, though there are few doubts in the validity of the conjecture at genus one. Much more interesting and yet obscure is the generalization to higher genera. This generalization is briefly discussed in s.3 at the end of the present paper.

2 Partition function

2.1 Method of calculation

To test the relation and find the correspondence between parameters, one needs a method of calculation of expressions J_k. We exploit the method of analytical continuation: to find a particular J_k, one expands the integrand (product of the theta-functions $\theta_*(z_i - z_j)$ and $\theta_*(z_i)$) in integer powers of q, and use the rule

$$\int_0^{2\pi} e^{iKz + iAz} dz = \frac{i(1 - e^{2\pi iA})}{A + K} \tag{8}$$

for integer K to integrate all the particular terms in the expansion. For natural N, β and μ, each finite order of the expansion contains a finite number of terms. Therefore, for natural values of parameters, this method easily allows one to find any particular J_k. Given these values at natural numbers, one can then restore the whole function by (the simplest variant of) analytical continuation: just assuming the dependence at non-natural values of parameters is given by a rational function. This method of calculation was suggested in [2], and we use it in the present paper as well.

2.2 Level 1

At level 1, for natural N, β and μ, one finds

$$J_1(A, \mu, \beta, N) = \Big[\beta^3 N^4 + (-5\beta^3 + 4\mu\beta^2 + 2\beta^2)N^3 + (7\beta^3 + \beta + 3\beta\mu^2 - 3A^2\beta + 6\beta\mu - 6\beta^2 - 12\mu\beta^2)N^2 + (-6A^2\mu -$$
$$2\mu^3 + 3A^2\beta + 8\mu\beta^2 - 6\beta\mu - \beta + 4\beta^2 - 3\beta^3 - 3\beta\mu^2 + 2\mu)N\Big]\Big[(A + \mu + \beta N - \beta + 1)(A - \mu - \beta N + \beta - 1)\Big]^{-1}$$

At particular value of $\mu = -\beta N$ this turns into

$$j_1(A, \beta, N) = \frac{\beta N(N+1)(2\beta^2 N^2 - 4\beta N + 2\beta^2 N + 4\beta + 3A^2 - 1 - 3\beta^2)}{(A - \beta + 1)(A + \beta - 1)} \tag{9}$$

Comparing this expression with eq. (2), one finds $B_1 = j_1 - \nu$ provided that

$$\Delta_{ext} = \beta N^2 + \beta N - N, \quad \Delta = \frac{A^2 - (\beta - 1)^2}{4\beta}, \quad \nu = 3\Delta_{ext} + 3N - 1 \tag{10}$$

This can be regarded as establishing the correspondence between parameters in (and simultaneously a check of) identity (6). Let us proceed to level 2, to further check relation (6).

2.3 Level 2

At level 2, the relation being tested, eq. (6) turns into

$$B_2 = j_2 - (3\Delta_{ext} + 3N - 1)(j_1 + 1) + \frac{(3\Delta_{ext} + 3N - 1)(3\Delta_{ext} + 3N - 2)}{2} \tag{11}$$

where B_2 and j_1 are given by (3) and (9), respectively. As can be deduced from general considerations, the function j_2 is a polynomial in N of degree 8; thus, to determine this polynomial unambiguously, one needs to calculate its values at 9 distinct natural values of N: say, from 1 to 9. Unfortunately, a direct computation of the partition function at these values of N is hardly feasible: the corresponding computer programs require too much time and memory for a successful run. For this reason, we select a slightly different way to test the relation at level 2: namely, we find j_2 from (11) and then test this *prediction* at many particular values of N and β.

Assuming the conventional dependence

$$c = 1 - 6\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^2 \tag{12}$$

the relation (11) implies the following *prediction* for j_2:

$$j_2^{(theor)} = \beta N(N+1)(-12 + 45A^2 - 76\beta N + 120\beta - 222A^2\beta + 94\beta^2 N^2 - 209\beta^3 N^2 + 366\beta^2 N - 745\beta^3 N +$$
$$176\beta^3 N^3 - 417\beta^2 + 678\beta^3 + 121A^2\beta N - 32\beta^2 N^3 - 4\beta N^2 - 48\beta^3 N^4 + 25\beta N^2 A^2 - 42A^4 - 561\beta^4 + 411A^2\beta^2 +$$
$$148\beta^4 N^4 - 480\beta^4 N^3 + 108\beta^4 N^2 + 864\beta^4 N - 136\beta^5 N^4 + 624\beta^5 N^3 + 39\beta^5 N^2 - 601\beta^5 N - 16N^6\beta^5 + 64N^5\beta^4 -$$
$$184N^5\beta^5 + 46N^4\beta^6 + 34N^6\beta^6 + 166N^5\beta^6 - 16N^6\beta^7 - 48N^5\beta^7 + 228\beta^5 + 331A^2\beta^3 N^2 + 559A^2\beta^3 N - 446A^2\beta^2 N -$$
$$366\beta^6 N^3 - 306A^2\beta^3 - 52\beta^6 N^2 - 198\beta^2 N^2 A^2 + 32\beta^4 N^4 A^2 + 232\beta^4 N^3 A^2 - 192\beta^3 N^3 A^2 - 134\beta^4 N^2 A^2 + 9A^6 -$$
$$36\beta^6 - 54A^4\beta^2 + 78A^4\beta + 81A^2\beta^4 + 104\beta^2 N^3 A^2 + 80\beta^7 N^3 + 12\beta^7 N^2 - 36N^4\beta^3 A^2 - 48\beta^5 N^4 A^2 - 116\beta^5 N^3 A^2 +$$
$$21\beta^5 N^2 A^2 + 12N^5\beta^5 A^2 - 42\beta^3 N^2 A^4 + 4\beta^5 N^6 A^2 - 16\beta^4 N^5 A^2 + 12\beta^3 N^4 A^4 + 24\beta^3 N^3 A^4 + 228\beta^6 N -$$
$$318\beta^4 N A^2 + 90A^4\beta^2 N + 66\beta^2 N^2 A^4 - 24\beta^2 N^3 A^4 - 66\beta N A^4 + 81\beta^5 A^2 N - 36\beta^7 N - 54\beta^3 A^4 N + 9N A^6\beta -$$
$$42\beta N^2 A^4 + 9\beta N^2 A^6)\Big[2(A - \beta + 1)(A + \beta - 1)(A - 1 + 2\beta)(A - 2 + \beta)(A + 2 - \beta)(A + 1 - 2\beta)\Big]^{-1}$$

Let us now check this prediction for particular values of N and β.

The case of $(N, \beta) = (1, 1)$

For natural μ one has:

$$J_2^{(exp)}\left(A, \mu, 1, 1\right) = \frac{\mu(-1 + 2\mu)(\mu^4 + 2\mu^3 + 5\mu^2 + 6\mu^2 A^2 + 16\mu - 18\mu A^2 + 12 - 33A^2 + 9A^4)}{(A + 1 + \mu)(A - 1 - \mu)(A - 2 - \mu)(A + 2 + \mu)} \tag{13}$$

Analytically continuing to $\mu = -\beta N = -1$, one finds

$$j_2^{(exp)}\left(A, 1, 1\right) = 27 = j_2^{(theor)}\left(A, 1, 1\right) \tag{14}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (1, 1)$.

The case of $(N, \beta) = (2, 1)$

For natural μ one has:

$$J_2^{(exp)}\left(A, \mu, 2, 1\right) = \Big(81A^2 - 54A^4 + 9A^6 - 36 + 174\mu A^2 - 408\mu A^4 + 54\mu A^6 + 84\mu + 968\mu^2 A^2 - 867\mu^2 A^4 +$$
$$72\mu^2 A^6 + 207\mu^2 + 1152\mu^3 A^2 - 402\mu^3 A^4 - 102\mu^3 + 395\mu^4 A^2 - 24\mu^4 A^4 - 242\mu^4 - 30\mu^5 A^2 + 24\mu^5 - 40\mu^6 A^2 +$$
$$79\mu^6 - 6\mu^7 - 8\mu^8\Big)\Big[(A + 2 + \mu)(A - 2 - \mu)(A - 3 - \mu)(A - 1 - \mu)(A + 1 + \mu)(A + 3 + \mu)\Big]^{-1}$$

Analytically continuing to $\mu = -\beta N = -2$, one finds

$$j_2^{(exp)}\left(A, 2, 1\right) = \frac{9(21A^4 + 35A^2 - 64)}{A^2(A^2 - 1)} = j_2^{(theor)}\left(A, 2, 1\right) \tag{15}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (2, 1)$.

The case of $(N, \beta) = (3, 1)$

For natural μ one has:

$$J_2^{(exp)}\left(A, \mu, 3, 1\right) = \Big(2700A^2 - 1755A^4 + 135A^6 - 1728\mu + 12888\mu A^2 - 5391\mu A^4 + 297\mu A^6 - 13356\mu^2 +$$
$$21942\mu^2 A^2 - 5157\mu^2 A^4 + 162\mu^2 A^6 - 16632\mu^3 + 12978\mu^3 A^2 - 1503\mu^3 A^4 - 5787\mu^4 + 2061\mu^4 A^2 - 54\mu^4 A^4 + 861\mu^5 -$$
$$333\mu^5 A^2 + 657\mu^6 - 90\mu^6 A^2 + 3\mu^7 - 18\mu^8\Big)\Big[(A + 3 + \mu)(A - 3 - \mu)(A - 4 - \mu)(A - 2 - \mu)(A + 2 + \mu)(A + 4 + \mu)\Big]^{-1}$$

Analytically continuing to $\mu = -\beta N = -3$, one finds

$$j_2^{(exp)}\left(A, 3, 1\right) = \frac{54(13A^6 + 78A^4 - 123A^2 + 192)}{A^2(A^2 - 1)^2} = j_2^{(theor)}\left(A, 3, 1\right) \tag{16}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (3, 1)$.

The case of $(N, \beta) = (1, 2)$

For natural μ one has:

$$J_2^{(exp)}\left(A, \mu, 1, 2\right) = \frac{\mu(-1 + 2\mu)(\mu^4 + 2\mu^3 + 5\mu^2 + 6\mu^2 A^2 + 16\mu - 18\mu A^2 + 12 - 33A^2 + 9A^4)}{(A + 1 + \mu)(A - 1 - \mu)(A - 2 - \mu)(A + 2 + \mu)} \tag{17}$$

Analytically continuing to $\mu = -\beta N = -2$, one finds

$$j_2^{(exp)}\left(A, 1, 2\right) = \frac{90(A^2 + 3)}{A^2 - 1} = j_2^{(theor)}\left(A, 1, 2\right) \tag{18}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (1, 2)$. In fact, the μ-dependent answer coincides completely with the $(N, \beta) = (1, 1)$ case, i.e. it does not depend on β. This property is specific for $N = 1$, where the Van-der-Monde factor actually does not contribute to the partition function.

The case of $(N, \beta) = (2, 2)$

For natural μ one has:

$$J_2^{(exp)}(A, \mu, 2, 2) = \Big(-576 + 846A^2 - 324A^4 + 54A^6 - 192\mu + 198\mu A^2 - 1332\mu A^4 + 126\mu A^6 + 1298\mu^2 +$$
$$2000\mu^2 A^2 - 1578\mu^2 A^4 + 72\mu^2 A^6 + 402\mu^3 + 2328\mu^3 A^2 - 522\mu^3 A^4 - 876\mu^4 + 722\mu^4 A^2 - 24\mu^4 A^4 - 228\mu^5 -$$
$$6\mu^5 A^2 + 162\mu^6 - 40\mu^6 A^2 + 18\mu^7 - 8\mu^8 \Big) \Big[(A+3+\mu)(A-3-\mu)(A-4-\mu)(A-1-\mu)(A+1+\mu)(A+4+\mu) \Big]^{-1}$$

Analytically continuing to $\mu = -\beta N = -4$, one finds

$$j_2^{(exp)}(A, 2, 2) = \frac{54(-2800 - 1663A^2 + 130A^4 + 13A^6)}{A^2(A^2-1)(A^2-9)} = j_2^{(theor)}(A, 2, 2) \tag{19}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (2, 2)$.

The case of $(N, \beta) = (3, 2)$

For natural μ one has:

$$J_2^{(exp)}(A, \mu, 3, 2) = \Big(-3240 + 36666A^2 - 13860A^4 + 594A^6 - 48276\mu + 121329\mu A^2 - 23994\mu A^4 + 621\mu A^6 - 201978\mu^2 +$$
$$133794\mu^2 A^2 - 13122\mu^2 A^4 + 162\mu^2 A^6 - 177237\mu^3 + 52956\mu^3 A^2 - 2367\mu^3 A^4 - 50814\mu^4 + 6174\mu^4 A^2 - 54\mu^4 A^4 - 66\mu^5 -$$
$$441\mu^5 A^2 + 1746\mu^6 - 90\mu^6 A^2 + 75\mu^7 - 18\mu^8 \Big) \Big[(A+5+\mu)(A-5-\mu)(A-6-\mu)(A-3-\mu)(A+3+\mu)(A+6+\mu) \Big]^{-1}$$

Analytically continuing to $\mu = -\beta N = -6$, one finds

$$j_2^{(exp)}(A, 3, 2) = \frac{180(15A^6 + 550A^4 - 453A^2 - 21168)}{A^2(A^2-1)^2(A^2-9)^2} = j_2^{(theor)}(A, 3, 2) \tag{20}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (3, 2)$.

The case of $(N, \beta) = (1, 3)$

For natural μ one has:

$$J_2^{(exp)}(A, \mu, 1, 3) = \frac{\mu(-1 + 2\mu)(\mu^4 + 2\mu^3 + 5\mu^2 + 6\mu^2 A^2 + 16\mu - 18\mu A^2 + 12 - 33A^2 + 9A^4)}{(A+1+\mu)(A-1-\mu)(A-2-\mu)(A+2+\mu)} \tag{21}$$

Analytically continuing to $\mu = -\beta N = -3$, one finds

$$j_2^{(exp)}(A, 1, 3) = \frac{63(3A^4 + 25A^2 + 12)}{(A^2-1)(A^2-4)} = j_2^{(theor)}(A, 1, 3) \tag{22}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (1, 3)$. Again, the μ-dependent answer coincides completely with the $(N, \beta) = (1, 1)$ case, because of β-independence of the partition function.

The case of $(N, \beta) = (2, 3)$

For natural μ one has:

$$J_2^{(exp)}(A, \mu, 2, 3) = \Big(-3600 + 4095A^2 - 630A^4 + 135A^6 - 3240\mu - 2334\mu A^2 - 2544\mu A^4 + 198\mu A^6 + 4705\mu^2 +$$
$$1880\mu^2 A^2 - 2397\mu^2 A^4 + 72\mu^2 A^6 + 3702\mu^3 + 3792\mu^3 A^2 - 642\mu^3 A^4 - 1666\mu^4 + 1157\mu^4 A^2 - 24\mu^4 A^4 - 864\mu^5 +$$
$$18\mu^5 A^2 + 209\mu^6 - 40\mu^6 A^2 + 42\mu^7 - 8\mu^8 \Big) \Big[(A+4+\mu)(A-4-\mu)(A-5-\mu)(A-1-\mu)(A+1+\mu)(A+5+\mu) \Big]^{-1}$$

Analytically continuing to $\mu = -\beta N = -6$, one finds

$$j_2^{(exp)}(A, 2, 3) = \frac{27(57A^6 + 1330A^4 - 45927A^2 - 425860)}{(A^2 - 1)(A^2 - 4)(A^2 - 25)} = j_2^{(theor)}(A, 2, 3) \tag{23}$$

This provides a check of eq. (6) at level 2 and in the case of $(N, \beta) = (2, 3)$. Together with the previous checks, this provides a rather firm evidence of validity of (6) at levels 1, 2 and at finite N, what is the main claim of the present paper.

3   Discussion

In this paper we explicitly demonstrated that if the Dotsenko-Fateev (DF) β-ensemble of [15, 1, 2] is defined as in refs.[15, 12] and the screening integration contour is chosen along the A-cycle, then it indeed reproduces the 1-point toric conformal block. We made an explicit check in the first two orders of q-expansion, but in this field this is well-known to be sufficient to get rid of any possible doubts. The two crucial features of this *new* DF representation for *generic* conformal blocks, which makes it different from the original DF formulas for minimal models and alike [16, 18], are
    (i) the use of a single (rather than two) screening charges, and
    (ii) the use of contour integrals along *open* (not obligatory/necessarilly *closed*) contours.
The procedure also includes an analytical continuation in α-parameters, but since every term of the q-expansion is a *rational* function of α's, this is not really a problem, as long as the conformal block is regarded as no more than a formal series (1) in q. The main difficulty at this level of consideration is specification of the integration contours.

   To understand the difference between the present paper and [15, 12], one should remember that there are three levels of accuracy in the definition of the DF β-ensembles [56].

   At the first level, one simply reproduces the Seiberg-Witten (SW) differential and the SW prepotential [57, 58, 59]. For this purpose one just needs to take the $g_s = \sqrt{-\epsilon_1 \epsilon_2} \to 0$ limit of the β-ensemble in the Dijkgraaf-Vafa (DV) phase [60], when the eigenvalues are concentrated within the cuts around extrema of the matrix-model potential W [61]. This limit is controlled by the quasiclassical approximation, it does not depend on the choice of integration contours (as soon as all extrema are within the integration domains), and it directly describes the SW prepotential as the β-ensemble free energy in terms of the spectral SW curve. The curve is seen already in the expression for the first resolvent $\rho^{(0|1)}(z) = \left\langle \text{tr} \frac{1}{\phi - z} \right\rangle$. Note that $\beta = b^2 = -\epsilon_1/\epsilon_2$ can already be arbitrary, only g_s is kept small. Most of the papers on matrix models and β-ensembles in the context of the AGT relation, including [12], are devoted to *this* level of consideration.

   At the next level, one restores all terms of the genus-expansion for the matrix model (β-ensemble) free energy. This can be most effectively done by the old resolvent techniques, which was recently revived under the name of the AMM-EO topological recursion [62]: as a general method of constructing the hierarchy of resolvents, implied by the Virasoro or W-constraints, starting from an arbitrary spectral curve defined as a covering. In application to the DV phases this method provides the free energy as a series in powers of g_s and rescaled multiplicities $S_\nu = g_s N_\nu$. In the original DV theory these series in S_q are also known as the CIV-DV prepotentials (with higher genus corrections). In application to the DF integrals this approach provides expressions which are exact in q-parameters.

   However, even this level of consideration is insufficient for the study of the AGT relations. The problem is that the conformal blocks are usually known in the form of q-series (1), but instead each coefficient $\mathcal{B}_k$ is a *rational* function of the dimensions, and, thus, of the multiplicities N_ν. This means that the CIV-DV series in S_ν should be exactly summed up before they can be compared with the known expressions for generic conformal blocks. In other words [56], from the point of view of the current AGT studies the topological recursion provides an excessive information about the q-dependencies, but insufficient information about the N_ν-dependencies. Therefore, at the third level of accuracy, one needs to specify *exactly* the integration contours in the DF integrals [1, 2], then these integrals can be *evaluated explicitly* – at least, after the q-expansion is performed and integrals belong to the (slightly extended) Euler-Selberg family [3]. Expanding the answers in powers of N_ν, one returns back to the CIV-DV potentials. This comparison of exact integrals and the CIV-DV expansions was explicitly performed in [56] for the standard simplest example of the 4-point conformal block, but there the guess of [2] was used for the proper choice of the contours. Unfortunately, there is still no clear idea of how the contours should be selected

for the conformal blocks on higher genus Riemann surfaces. In the present paper, we checked the new *guess* of [15, 12]: that for $g = 1$ an additional contour should be the A-cycle, provided the matrix model potential is **modified** *al la* **[15], by adding the shift along the Jacobian**.

A motivation for this shift comes from calculating the free parameters in the AGT relation. A genus g conformal block with n external lines (punctures) has $3g - 3 + 2n$ free parameters: the dimensions of fields on n external and $3g - 3 + n$ internal lines of the corresponding Feynman diagram. This same number of parameters should be of course present in the DF representation of the conformal block. At the *first* level in above classification, this means that the number P of parameters in the matrix model potential $W(z)$ and the number Z of zeroes of $dW(z)$ should sum up to

$$P + Z - 1 = 3g - 3 + 2n \tag{24}$$

Unity is subtracted from the l.h.s. because of the "conservation law"

$$\left(\sum_{i=1}^{n} \alpha_i + b \sum_{\nu=1}^{2g+n-2} N_\nu \right) = \left(\frac{1}{b} - b \right)(g - 1) \tag{25}$$

which is a characteristic feature of the DF integrals. The number of zeroes of the differential with n poles on the genus g Riemann surface (not obligatory single-valued) is given by the Riemann-Roch theorem: it is $Z = 2g+n-2$. Thus, P should be equal to $P = n + g$. Of these n are the α-parameters of the n external lines, and g should be looked for somewhere else. A suggestion of [15] was simply to add an arbitrary linear combination of holomorphic differentials to $dW(z)$:

$$dW = 2b \sum_{i=1}^{n} \alpha_i d \log E(z, x_i) + \sum_{k=1}^{g} p_k \omega_k \tag{26}$$

where $E(z, x) = \frac{\theta_*(\vec{z} - \vec{x})}{\nu_*(z)\nu_*(x)}$ is the prime-form, x's are positions of the n punctures and ω_k are the g holomorphic differentials. The $n + g$ free parameters are α_i and p_k. In [12] it was checked that, for $g = 1$, this prescription reproduces the relevant SW curve, and in the present paper we checked much more: that the relevant conformal block is also reproduced.

For this, however, we had to specify the DF integral in more details. In the DF representation, the multiplicities N_ν are the multiplicities of different integrals of the screening operator $V_b = e^{ib\phi}$, and to fully define the integral one should explicitly specify the $\#(\nu) = Z = 2g + n - 2$ integration contours. According to [2], for genus $g = 0$ these $n - 2$ contours connect the pairs of external lines: the first and the second, the first and the third and so on, up to the $(n-1)$-st one, while the n-th puncture is always at infinity: $dW^{(g=0)} = \sum_{i=1}^{n-1} \frac{2b\alpha_i}{z - x_i} \, dz$ and α_n is defined from the conservation law (25). For $g > 0$ the choice of contours is far more obscure. For $g = 1$ the suggestion of [15, 12] seems to be: $n - 1$ contour between the pairs of the punctures and one additional contour along the A-cycle. In this paper we checked that this indeed works for $n = 1$ and there are few doubts that this will be true for all $n > 1$, though this remains to be checked as well. Unfortunately, this sheds no light on possible choices of contours for $g \geq 2$: there are still $n - 1$ "obvious" contours between the punctures, but it is not very clear what the remaining $2g - 1$ are going to be. This puzzle remains to be resolved.

It deserves noting that from the point of view of DF representation there could be another possibility, mentioned in [2]. Instead of adding p_k parameters to (26), one could simply choose $3g - 1$ additional contours, so that multiplicities of the corresponding integrations complemented the n external α-parameters and $n - 1$ multiplicities of "obvious contours" between the punctures to match the number of parameters in conformal block:

$$3g - 1 + n + (n - 1) - 1 = 3g + 2n - 3 \tag{27}$$

As usual, unity is subtracted from the l.h.s. because of the conservation law (25), where sum over ν this times goes from 1 to $3g + n - 2$. The point of [2] was that $3g - 1$ is a nice number: it is the quantity of *non-homotopic* closed contours on the genus g Riemann surface (which for $g > 1$ exceeds the number $2g$ of non-homological A and B cycles). Unfortunately, we did not manage to make this prescription working even for $(g, n) = (1, 1)$, what, perhaps, is not a surprise because it was not made consistent with the first (quasiclassical) level of the DV description. Still, while the problem of contour choice remains open, one should not full neglect this alternative possibility. (For example, one can use all the non-homotopic contours besides B-cycles, getting $2g - 1 = 3g - 1 - g$ closed contours...)

Of course, for any consideration of the AGT relations at higher genera, $g > 1$, one should also deduce, at least, the first terms of q-expansions of higher genus conformal blocks from representation theory. This is a straightforward, but tedious calculation of its own value, which is not yet reported in the literature.

Acknowledgements

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